Radiation from a Rectangular Waveguide Filled with Ferrite*

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Summary—This paper presents an approximate analytical solution to the problem of radiation from a ferrite-filled rectangular waveguide. The field distribution at the mouth of the guide is assumed to be unaffected by the termination of the guide. The vector Huygens' principle is applied to find the far-zone radiation field from the determined aperture field.

The solution to the problem is found in this manner for the cases of longitudinal and transverse magnetization of the ferrite. The transverse magnetization case is supplemented with a discussion of a specific numerical example which includes plots of the aperture field distribution and the phase angle as well as plots of the far-zone radiation field. The experimentally known phenomenon of the effect of the applied magnetic field upon the shift of the main lobe[1] is demonstrated and verified analytically.

INTRODUCTION

THE low-loss ferrite medium has been actively investigated during the past few years. Polder[2] worked out the effective properties of this medium for plane waves, and Hogan[3] has made various experimental studies of the propagation in a cylindrical guide containing ferrites. Gintsburg,[4] and Suhl and Walker[5] solved the problem of a circular waveguide completely filled with a longitudinally magnetized ferrite medium. Angelakos and Korman[1] made measurements of the radiation pattern from an open-end rectangular waveguide completely filled with a transversely magnetized ferrite.

The nature of propagation of electromagnetic energy through a magnetized ferrite medium is described by the Maxwell's equations which connect the space variations of $\boldsymbol{E}$ and $\boldsymbol{H}$ with the time variations of $\boldsymbol{D}$ and $\boldsymbol{B}$. The relation between the magnetic intensity vector, $\boldsymbol{H}$, and the magnetic induction vector, $\boldsymbol{B}$, in a ferrite medium is characterized in the form $\boldsymbol{B}=(\mu)\boldsymbol{H}$ where (μ) is the permeability tensor. The components of the permeability tensor are derived from the mathematical model which assumes a ferrite sample fully saturated in dc magnetic fields. The other assumptions made are that ac quantities in the equation of motion of the magnetization are small in comparison with the dc quantities, so that their products can be neglected and only linear terms be considered, and also that the ferrite is lossfree.

With these assumptions the permeability tensor is written in the form[2]

$$(\mu) = \begin{bmatrix} \mu & -j\kappa & 0 \\ j\kappa & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix} \tag{1}$$

when the applied magnetic field is along the direction of propagation (z axis) and

$$(\mu) = \begin{bmatrix} \mu & 0 & -j\kappa \\ 0 & \mu_0 & 0 \\ j\kappa & 0 & \mu \end{bmatrix} \tag{2}$$

when the direction of the applied magnetic field is transverse (along the y axis) to the direction of propagation. The components of the permeability tensor are expressed in terms of the applied magnetic field as follows:

$$\mu = \mu_0\left(1 + \frac{p\sigma}{\sigma^2 - 1}\right)$$
$$\kappa = \mu_0\left(\frac{p}{\sigma^2 - 1}\right) \tag{3}$$

where

$$\sigma = \frac{|\gamma| H_0}{\omega}; \qquad p = \frac{|\gamma| M_0}{\mu_0\omega} \tag{4}$$

and

μ_0 = permeability of free space,
γ = magnetomechanical ratio as classified by Darrow,[6]
H_0 = applied magnetostatic field,
M_0 = dc magnetization (resulting from and directed along H_0).

The notations σ and p have been adopted in an unaltered form from Suhl and Walker.[5] It should be noted at this point that σ represents the ratio of natural precession frequency $|\gamma|H_0/2\pi$ to the carrier frequency and p is the ratio of a frequency $|\gamma|M_0/2\pi\mu_0$, associated with the saturation magnetization M_0, to the carrier

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[1] D. J. Angelakos and M. M. Korman, "Radiation from ferrite-filled apertures," PROC. IRE, vol. 44, pp. 1463–1468; October, 1956.
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[5] H. Suhl and L. R. Walker, "Topics in guided-wave propagation through gyromagnetic media, Part I—The completely filled cylindrical guide," *Bell Sys. Tech. J.*, vol. 33, pp. 579–597; May, 1954.
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frequency. It is important to observe that σ and p always have the same signs; *i.e.*, when H_0 is reversed, so is the saturation magnetization. Furthermore, μ is an even function of the magnetic field whereas κ is an odd function.

Field Components in a Parallel Plane Waveguide

Longitudinal Magnetization

The case of a rectangular waveguide completely filled with a ferrite that is subjected to a steady magnetic field in the direction of propagation has been treated by several authors.[7–9] Chambers used perturbation methods to show the existence of quasi TE and TM modes and he calculated the first term of the power expansion for the fields. Suhl and Walker[8] treated the case of a limit TEM mode in a parallel plane guide. Van Trier[9] discussed the nature of the various modes that can exist in a parallel plane guide.

In this paper we shall derive explicit expressions for the various field components and the propagation constant for the case of a parallel plane waveguide. We begin with the equations derived by Kales[10] from the two Maxwell's curl-equations and the relation of (1)

$$\begin{aligned} \nabla_t^2 E_z + (k_2^2\chi - \beta^2)E_z &= -j\beta \frac{\omega\kappa\mu_0}{\mu} H_z \\ \nabla_t^2 H_z + \left(k_2^2 - \frac{\mu_0}{\mu}\cdot\beta^2\right)H_z &= j\beta \frac{\omega\epsilon\kappa}{\mu} E_z \end{aligned} \tag{5}$$

where

$$k_2 = \omega\sqrt{\epsilon\mu_0}$$

$$\chi = \frac{\mu}{\mu_0}\left(1 - \frac{\kappa^2}{\mu^2}\right)$$

and ϵ = electric inductive capacity[11] for the ferrite. The time and z-dependence of the form $\epsilon^{j(\omega t - \beta z)}$ has been assumed.

It has been shown[7–10] that fields satisfying (5) cannot be separated into the conventional TE, TM, and TEM modes. We shall, therefore, look for any unconventional mode that can propagate under these conditions.

We have to obtain an equation in terms of H_z or E_z only. This can be done by elimination of H_z and $\nabla_t^2 H_z$ or E_z and $\nabla_t^2 E_z$ between the equations of (5). Since it is more convenient to apply the boundary conditions to the electric field, we shall solve (5) for E_z, getting:

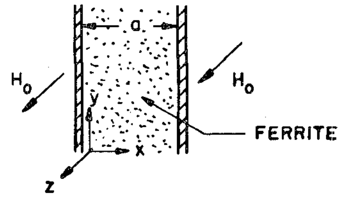

Fig. 1—Geometry of a parallel plane waveguide.

$$\nabla_t^4 E_z + \Psi\nabla_t^2 E_z + \Upsilon^2 E_z = 0 \tag{6}$$

where

$$\Psi = (\chi + 1)k_2^2 - \left(\frac{\mu_0}{\mu} + 1\right)\beta^2 \tag{7}$$

$$\Upsilon = \left[\chi k_2^4 - 2\,k_2^2\beta^2 + \frac{\mu_0}{\mu}\beta^4\right]^{1/2}. \tag{8}$$

Eq. (6) is a fourth-order partial differential equation which cannot be solved in the Cartesian coordinate system by the method of separation of variables. Solution is possible, however, in the case of a parallel plane waveguide where $\partial/\partial y = 0$. In such a case we get

$$\frac{d^4E_z}{dx^4} + \Psi\frac{d^2E_z}{dx^2} + \Upsilon^2 E_z = 0. \tag{9}$$

The other field components can be found to be

$$E_x = j\beta\int E_z dx \tag{10}$$

$$E_y = \frac{\mu}{\beta\kappa}\left[\frac{dE_z}{dx} + M\int E_z dx\right] \tag{11}$$

$$H_x = -\frac{1}{\omega\kappa}\left[\frac{dE_z}{dx} + N\int E_z dx\right] \tag{12}$$

$$H_y = j\omega\epsilon\int E_z dx \tag{13}$$

$$H_z = \frac{j\mu}{\beta\omega\kappa\mu_0}\left[\frac{d^2E_z}{dx^2} + ME_z\right] \tag{14}$$

where

$$M = \chi k_2^2 - \beta^2; \qquad N = \frac{\mu}{\mu_0}k_2^2 - \beta^2. \tag{15}$$

Eq. (9) describes the longitudinal component of the electric field in a longitudinally magnetized ferrite medium between two parallel planes. We shall choose the coordinate system as shown in Fig. 1.

The appropriate boundary conditions are

$$E_z = 0$$

$$\frac{dE_z}{dx} + M\int E_z dx = 0 \tag{16}$$

on the boundaries $x = 0$ and $x = a$.

A solution to (9) will depend upon whether the quantity $(\Psi/2)^2 - \Upsilon^2$ is positive or negative. By going

[7] L. G. Chambers, "Propagation in a ferrite-filled waveguide," *Quart. J. Mech. Appl. Math.*, vol. 8, pp. 435–447; December, 1955.

[8] H. Suhl and L. R. Walker, "Topics in guided-wave propagation through gyromagnetic media, Part III—Perturbation theory and miscellaneous results," *Bell Sys. Tech. J.*, vol. 33, pp. 1133–1194; September, 1954.

[9] A. A. Van Trier, "Guided electromagnetic waves in anisotropic media," *Applied Sci. Res.*, sec. B, vol. 3, no. 4, 5; 1953.

[10] M. L. Kales, "Modes in waveguides containing ferrites," *J. Appl. Phys.*, vol. 24, pp. 604–608; May, 1953.

[11] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Co., Inc., New York, N. Y., p. 10, 1941.

back to the original definitions of μ, κ, and χ, it can be shown that

$$\left(\frac{\Psi}{2}\right)^2 - \Upsilon^2 = \frac{p^2}{4(1-\sigma p - \sigma^2)^2} \cdot \left\{ [(\sigma + p)k_2^2 - \sigma\beta^2]^2 + 4k_2^2\beta^2 \right\}, \quad (17)$$

hence the expression is a positive definite. Therefore we have to consider one case only, namely

$$\left(\frac{\Psi}{2}\right)^2 - \Upsilon^2 > 0.$$

A general solution to (9) can now be written in the form

$$E_z = C_1 \cos \rho_1 x + C_2 \sin \rho_1 x + C_3 \cos \rho_2 x + C_4 \sin \rho_2 x \quad (18)$$

where

$$\rho_{1,2} = \sqrt{\frac{\Psi}{2} \pm \sqrt{\left(\frac{\Psi}{2}\right)^2 - \Upsilon^2}}. \quad (19)$$

After the boundary conditions of (16) are evaluated and the determinant of the coefficients is set equal to zero to obtain a unique solution for the constants C_1 through C_4, the following condition is obtained

$$2\left(\rho_1 - \frac{M}{\rho^1}\right)\left(\rho_2 - \frac{M}{\rho_2}\right)(1 - \cos \rho_1 a \cos \rho_2 a) = \left[\left(\rho_1 - \frac{M}{\rho_1}\right)^2 + \left(\rho_2 - \frac{M}{\rho_2}\right)^2\right] \sin \rho_1 a \sin \rho_2 a. \quad (20)$$

The above relation will be satisfied if we choose ρ_1 and ρ_2 such that

$$\rho_1 = \frac{m\pi}{a}; \qquad \rho_2 = \frac{n\pi}{a} \quad (21)$$

where m and n are two integers,[12] $m \neq n$, both simultaneously odd or even. As a result of this, the constants separate into two independent groups, namely

$$C_3 = -C_1 \quad (22)$$

and

$$C_4 = -\frac{\rho_2}{\rho_1}\left(\frac{\rho_1^2 - M}{\rho_2^2 - M}\right) C_2. \quad (23)$$

Evidently the solution can be written in one of the two ways

$$E_z = C(\cos \rho_1 x - \cos \rho_2 x) e^{j(\omega t - \beta z)} \quad (24)$$

or

$$E_z = C\left(\sin \rho_1 x - \frac{\rho_2(\rho_1^2 - M)}{\rho_1(\rho_2^2 - M)} \sin \rho_2 x\right) e^{j(\omega t - \beta z)}. \quad (25)$$

Since these two solutions are independent of each other, this seems to suggest that the modes in the ferrite can be separated into two independent groups. It will be shown later that the first group represented by (24) goes over to the usual form of TE mode and the second group represented by (25) goes over to the usual form of TM mode when the external magnetostatic field is removed. We shall, therefore, call the first group a quasi TE mode and the second group a quasi TM mode which is in agreement with the conclusions arrived at by other authors.[4,5,7,9]

The remaining field components can now be written as follows:

Quasi TE mode

$$E_x = j\beta C \left\{ \frac{\sin \rho_1 x}{\rho_1} - \frac{\sin \rho_2 x}{\rho_2} \right\} \quad (26)$$

$$E_y = -\frac{\mu}{\beta\kappa} C \left\{ (\rho_1^2 - M)\frac{\sin \rho_1 x}{\rho_1} - (\rho_2^2 - M)\frac{\sin \rho_2 x}{\rho_2} \right\} \quad (27)$$

$$H_x = \frac{C}{\kappa\omega} \left\{ (\rho_1^2 - N)\frac{\sin \rho_1 x}{\rho_1} - (\rho_2^2 - N)\frac{\sin \rho_2 x}{\rho_2} \right\} \quad (28)$$

$$H_y = j\omega\epsilon C \left\{ \frac{\sin \rho_1 x}{\rho_1} - \frac{\sin \rho_2 x}{\rho_2} \right\} \quad (29)$$

$$H_z = \frac{\mu C}{j\beta\omega\kappa\mu_0} \left\{ (\rho_1^2 - M)\cos \rho_1 x - (\rho_2^2 - M)\cos \rho_2 x \right\}. \quad (30)$$

Quasi TM mode

$$E_x = \frac{j\beta}{\rho_1} C \left\{ \cos \rho_1 x - \left(\frac{\rho_1^2 - M}{\rho_2^2 - M}\right) \cos \rho_2 x \right\} \quad (31)$$

$$E_y = \frac{\mu(\rho_1^2 - M)}{\beta\kappa\rho_1} C\{\cos \rho_1 x - \cos \rho_2 x\} \quad (32)$$

$$H_x = -\frac{(\rho_2^2 - N)}{\omega\kappa\rho_1} C \left\{ \left(\frac{\rho_1^2 - N}{\rho_2^2 - N}\right) \cdot \cos \rho_1 x - \left(\frac{\rho_1^2 - M}{\rho_2^2 + M}\right) \cos \rho_2 x \right\} \quad (33)$$

$$H_y = -\frac{j\omega\epsilon}{\rho_1} C \left\{ \cos \rho_1 x - \left(\frac{\rho_1^2 - M}{\rho_2^2 - M}\right) \cos \rho_2 x \right\} \quad (34)$$

$$H_z = -\frac{j\mu(\rho_1^2 - M)}{\beta\omega\kappa\mu_0} C \left\{ \sin \rho_1 x - \frac{\rho_2}{\rho_1} \sin \rho_2 x \right\}. \quad (35)$$

The propagation constant can be found easily if we notice the following useful relationships:

$$\rho_1\rho_2 = \frac{mn\pi^2}{a^2} = \Upsilon \quad (36)$$

$$\rho_1^2 + \rho_2^2 = (m^2 + n^2)\frac{\pi^2}{a^2} = \Psi. \quad (37)$$

From (37) and (7) it follows immediately that

$$\beta^2 = \frac{1}{1 + \frac{\mu_0}{\mu}} \left\{ (\chi + 1)k_2^2 - (m^2 + n^2)\frac{\pi^2}{a^2} \right\}. \quad (38)$$

A graph of the parameter x is shown in fig. 2.

Until the present time little has been said about the

[12] The same conclusion can be drawn from the relations derived by Van Trier, *op. cit.*, p. 327 (2.57) and (2.58).

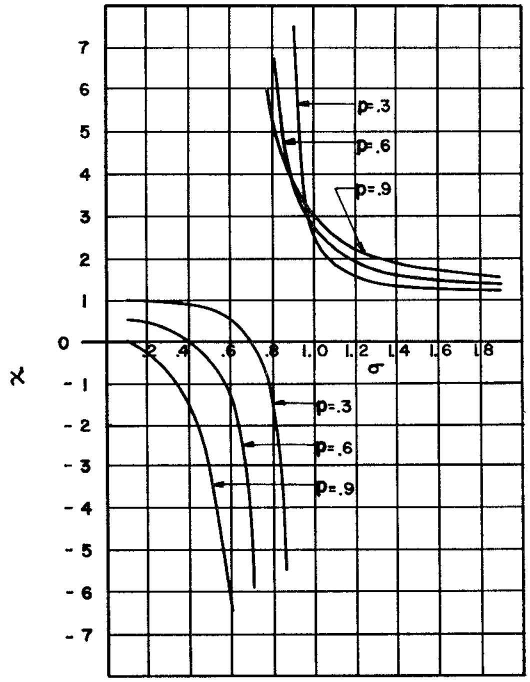

Fig. 2—Relation between χ and σ.

integers m and n except that the boundary conditions impose a restriction that they both be simultaneously odd or even. The other restrictions to be considered are those imposed by the medium itself. They are expressed formally in (36) and (37). Thus, as it could be expected, m and n are not independent; *i.e.*, once one of them is chosen arbitrarily the other one is fixed, and vice versa. The relation connecting the integers m and n can be found if we first eliminate β between (7) and (8) which yields

$$\frac{\Psi}{\omega^2\epsilon\kappa} = -\frac{\kappa}{\mu} \pm \left(1+\frac{\mu_0}{\mu}\right)\sqrt{1+\frac{\mu}{\mu_0}\left(\frac{\Upsilon}{\omega^2\epsilon\kappa}\right)^2}. \quad (39)$$

We can now substitute the relations of (36) and (37) to obtain the desired relationship

$$\rho_2^2 = \left(\frac{\mu^2+\mu_0^2}{2\mu\mu_0}\right)\rho_1^2 - \frac{\omega^2\epsilon\kappa^2}{\mu} \pm \left\{\left(\frac{\mu^2-\mu_0^2}{2\mu\mu_0}\right)^2\rho_1^4 + (\mu+\mu_0)^2\left(\omega^2\epsilon - \frac{\rho_1^2}{\mu_0}\right)\frac{\omega^2\epsilon\kappa^2}{\mu^2}\right\}^{1/2} \quad (40)$$

or

$$n^2 = \left(\frac{\mu^2+\mu_0^2}{2\mu\mu_0}\right)m^2 - \frac{4a^2\epsilon'\kappa^2}{\mu\mu_0\lambda^2} \pm \frac{1}{\mu\mu_0}\left\{\left(\frac{\mu^2-\mu_0^2}{2}\right)^2 m^4 + (\mu+\mu_0)^2\left(\frac{4a^2\epsilon'}{\lambda^2} - m^2\right)\frac{4a^2\epsilon'\kappa^2}{\lambda^2}\right\}^{1/2} \quad (41)$$

where $\epsilon' = \epsilon/\epsilon_0$ and λ is the free space wavelength. Eq. (41) is just about the simplest form of expressing the relation between m and n. It can be solved either graphically or by means of a digital computer.

It is interesting to find out what happens to our solutions as we pass to the limit of zero applied magnetostatic field. When $\sigma = 0$ we can see from (3)

$$\mu = \mu_0$$
$$\kappa = \lim_{p\to 0}(p\mu_0). \quad (42)$$

The propagation constant becomes

$$\beta^2 = \lim_{p\to 0}\left\{k_2^2 - \frac{\rho_1^2+\rho_2^2}{2} - \frac{p^2k_2^2}{2}\right\} = k_2^2 - \frac{\rho_1^2+\rho_2^2}{2}. \quad (43)$$

Evidently when $\sigma = 0$, $\rho_1 = \rho_2$ for β to take on the usual form. This can be also shown to be true if we put $\mu = \mu_0$, $\kappa = \lim_{p\to 0}(p\mu_0)$ into (40). We obtain

$$\rho_1^2 - \rho_2^2 = \lim_{p\to 0}\left\{2pk_2\sqrt{k_2^2 - \frac{\rho_1^2+\rho_2^2}{2} - \frac{k_2^2p^2}{4}}\right\} \quad (44)$$

which can also be written as

$$\rho_1^2 - \rho_2^2 = \lim_{p\to 0}\left\{2pk_2\beta\sqrt{1+\frac{k_2^2p^2}{4\beta^2}}\right\} = 0. \quad (45)$$

Thus our propagation constant reduces properly to the form it should have when the external magnetostatic field is removed and the medium becomes isotropic. Next, let us investigate the field components. We can see immediately that when $\rho_1 = \rho_2$ in the quasi TE group then E_z, E_x, and H_y become zero. In the case of the remaining field components we have to exercise a little care going over to the limit since the expressions become indeterminate if we simply put $\rho_1 = \rho_2$ and $\kappa = 0$. If we can imagine a weak enough magnetostatic field such that we can set

$$\frac{\sin\rho_1 x}{\rho_1} \approx \frac{\sin\rho_2 x}{\rho_2}$$
$$\cos\rho_1 x \approx \cos\rho_2 x$$
$$\rho_1^2 \neq \rho_2^2,$$

then we would have for a typical field component, say E_y,

$$E_y = \frac{C}{\beta p\rho_1}(\rho_1^2 - \rho_2^2)\sin\rho_1 x.$$

Substituting for $(\rho_1^2 - \rho_2^2)$ the expression obtained in (45) we get

$$E_y = \frac{2Ck_2}{\rho_1}\sqrt{1+\frac{k_2^2p^2}{4\beta^2}}\sin\rho_1 x.$$

Finally, as we pass to the limit $p = 0$ this becomes simply

$$E_y = K\sin\rho x \quad (46)$$

where K is a constant. The other nonzero field components become

$$H_x = -\frac{\beta}{\omega\mu_0} K \sin \rho x \quad (47)$$

$$H_z = j\frac{\rho}{\omega\mu_0} K \cos \rho x. \quad (48)$$

These equations can be recognized as those describing a TE wave between two parallel planes. Thus our solution in the ferrite medium that we called quasi TE group goes over properly to a TE mode when the external magnetostatic field is removed.

By a similar procedure of going over to the limit of zero magnetostatic field it can be easily shown that the quasi TM group takes on the usual form of TM mode in an isotropic medium.

Transverse Magnetization

The case of a rectangular waveguide containing a ferrite medium subjected to a magnetostatic field transverse in relation to the direction of propagation has been treated by many authors.[9,13–15] Van Trier[9] has solved the case of TE_{n0} mode in a transversely magnetized parallel plane waveguide. Chevalier and Polacco[13] and Epstein[14] also worked on the same problem. Vartanian and Jaynes[15] worked out a solution to the problem of propagation of higher order modes in a transversely magnetized ferrite-filled rectangular waveguide.

In this paper we shall be concerned with a ferrite-filled transversely magnetized parallel plane waveguide that can propagate more than one mode at a time. We shall start with the expressions for the field components and the propagation constant for a single mode guide as found elsewhere[9]

$$E_y = C \sin \nu x \quad (49)$$

$$H_x = -\frac{C}{\omega\mu_0\chi}\left(\frac{\kappa\nu}{\mu}\cos \nu x + \beta \sin \nu x\right) \quad (50)$$

$$H_z = \frac{jC}{\omega\mu_0\chi}(\nu \cos \nu x + \kappa \sin \nu x) \quad (51)$$

$$\beta^2 = k_2{}^2\chi - \nu^2; \qquad \nu = \frac{m\pi}{a}. \quad (52)$$

It is of interest to examine the above equations more carefully. The expression for the electric field is identical with that of a conventional TE_{m0} mode in a rectangular waveguide filled with an isotropic medium. The magnetic field components have additional terms which are functions of the applied magnetostatic field. The propagation constant and the cutoff wavelength deserve special attention. Because of the presence of the parameter χ, the cutoff wavelength (and by the same token the propagation constant) can be adjusted over wide limits. Thus, theoretically, we can make the waveguide "electrically" as large or as small as we please. In Fig. 3 there is a plot of the cutoff wavelength for TE_{10} and TE_{20} modes as a function of the applied magnetostatic field for a standard size waveguide of 0.5×1.0 inch where the dielectric constant of the ferrite has been arbitrarily set equal to one.

In the next section we shall be concerned with one of the possible applications of the electrically adjustable propagation constant, *i.e.*, "electrical scanning." Let us consider waveguide which is terminated with a transversely magnetized ferrite plate as shown in Fig. 4.

We are interested at this moment in the expression for the transverse components of the electric and magnetic fields existing at the mouth of the waveguide. This problem is solved approximately by assuming that the fields at the mouth of the guide are the same as those that would exist were the waveguide not terminated there. To the left of the ferrite plate we assume the existence of the fundamental mode only, *i.e.*, TE_{10} mode. In the ferrite region the waveguide width can be made electrically large by properly adjusting the external magnetostatic field H_0 so that higher order modes can propagate. We assume that some of the incident power is converted into higher order modes at the surface of the ferrite and, since the waveguide is large enough electrically, they may propagate unattenuated. Let l be the thickness of the ferrite plate. The transverse fields at the mouth of the guide can then be written:

$$E_y = C_1 e^{-j\beta_1 l}\sum_{m=1}^{M}\frac{C_m}{C_1}\sin\frac{m\pi x}{a}e^{j\delta_m} \quad (53)$$

$$H_x = -\frac{C_1 e^{j\beta_1 l}}{\omega\mu_0\chi}\sum_{m=1}^{M}\frac{C_m}{C_1}\left(\frac{\kappa}{\mu}\frac{m\pi}{a}\cos\frac{m\pi x}{a} + \beta_m \sin\frac{m\pi x}{a}\right)e^{j\delta_m} \quad (54)$$

where

$$\delta_m = (\beta_1 - \beta_m)\iota. \quad (55)$$

The term C_m/C_1, which is the ratio of the amplitude of the electric field in the mth mode to that in the principal mode, is an unknown. Angelakos and Korman[1] concluded from their experiments with a ferrite-loaded waveguide that some of the incident power is converted into higher order modes upon entering the ferrite region but they made no measurements of the ratio of mode conversion. There is no analytical solution to the problem either. Some of the difficulties connected with obtaining a workable analytical expression have been out-

[13] A. Chevalier and E. Polacco, "Propagation of electromagnetic TE wave in a guide containing ferrites," *C. R. Acad. Sci.*, vol. 239, pp. 692–694; September 20, 1954.

[14] P. S. Epstein, "Wave propagation in a gyromagnetic medium," *Rev. Mod. Phys.*, vol. 28, pp. 3–17; January, 1956.

[15] P. H. Vartanian and E. T. Jaynes, "Propagation in ferrite-filled transversely magnetized waveguide," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-4, pp. 140–143; July, 1956.

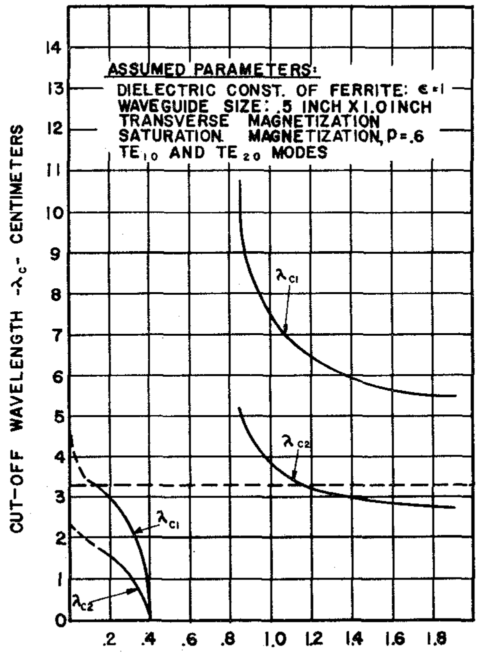

Fig. 3—Cutoff wavelength vs magnetostatic field.

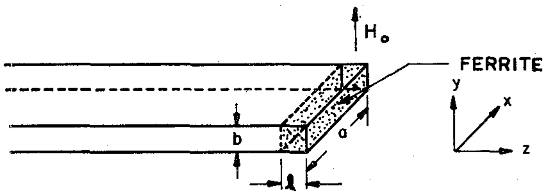

Fig. 4—Waveguide and ferrite plate configuration for solution of the problem of "electrical scanning."

lined by Epstein.[14] In the next section, when computing the radiation field, certain arbitrary values will be chosen for this ratio.

A few words should be mentioned about reciprocity. From (52), it is evident that the system is reciprocal as far as the propagation constant is concerned. Let us, however, examine the wave admittances. We know that in a reciprocal system the wave admittance of the incident wave is equal to that of the reflected wave. Now let us examine the wave admittances for the ferrite. For the incident wave we obtain

$$Y_{yx}^{+} = -\frac{H_x^{+}}{E_y^{+}} = \frac{1}{\omega\mu_0\chi}\sum_{m=1}^{M}\left(\beta_m + \frac{\kappa}{\mu}\frac{m\pi}{a}\cot\frac{m\pi x}{a}\right) \quad (56)$$

and for the reflected wave we obtain

$$Y_{yx}^{-} = \frac{H_x^{-}}{E_y^{-}} = \frac{1}{\omega\mu_0\chi}\sum_{m=1}^{M}\left(\beta_m - \frac{\kappa}{\mu}\frac{m\pi}{a}\cot\frac{m\pi x}{a}\right). \quad (57)$$

Thus, the system is not reciprocal! The forward and backward wave admittances for a waveguide able to propagate two modes are plotted in Fig. 5. It can be seen that these admittances have a mirror symmetry

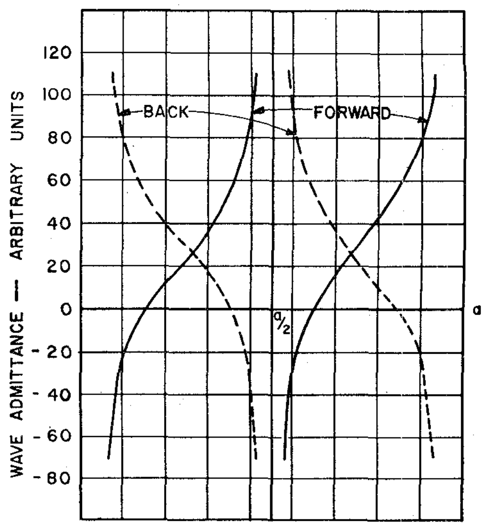

Fig. 5—Aperture forward and back wave admittances.

about the center of the guide. It can be shown that if the direction of the magnetostatic field is changed while the incident wave is being received, the input and output admittances are identical and the reciprocity is restored. The implications of this fact will be considered when discussing the reciprocity of radiation patterns.

Radiation from a Rectangular Waveguide Filled with Ferrite

Introduction to the Problem

The problem of radiation from an open-end waveguide has not, thus far, been solved rigorously. Actually the radiation should be considered to arise from the current distribution on the inside walls of the guide, which is just the current distribution associated with the fields propagated in the interior of the guide, together with the currents flowing from the open end upon the exterior guide surface. Due to this complexity a rigorous solution to the radiation problem has not yet been found. The problem has, however, been solved approximately.[16] In this method the guide opening is assumed to act as an aperture in an infinite screen, and the transverse fields in the aperture are assumed to be the same as those that would exist if the guide did not terminate there. The vector Huygens' formula is applied to obtain the radiation field from the aperture field distribution as discussed by Silver.[17]

We follow this method in the solution of the problem of radiation from an open-end waveguide that is completely filled with ferrite. The expressions for the far-zone fields as derived for a waveguide containing an isotropic medium are modified to suit our purpose. The

[16] L. J. Chu, "Calculation of the radiation properties of hollow pipes and horns," *J. Appl. Phys.*, vol. 11, pp. 603–610; September 1940.

[17] S. Silver, "Microwave Antenna Theory and Design," McGraw Hill Book Co., Inc., New York, N. Y., pp. 158–162; 1949.

modified equations are: in the H plane ($\phi=0$)

$$E_\theta = \frac{jke^{-jkR}}{4\pi R}\int_{\text{c.s.}} (E_x + \eta\cos\theta H_y)e^{jkx\sin\theta}dS \quad (58)$$

$$E_\phi = \frac{jke^{-jkR}}{4\pi R}\int_{\text{c.s.}} (\cos\theta E_y - \eta H_x)e^{jkx\sin\theta}dS, \quad (59)$$

and in the E plane ($\varphi=\pi/2$)

$$E_\theta = \frac{jke^{-jkR}}{4\pi R}\int_{\text{c.s.}} (E_y - \eta\cos\theta H_x)e^{jky\sin\theta}dS \quad (60)$$

$$E_\phi = -\frac{jke^{-jkR}}{4\pi R}\int_{\text{c.s.}} (\cos\theta E_x + \eta H_y)e^{jky\sin\theta}dS \quad (61)$$

where $k=\omega\sqrt{\mu_0\epsilon_0}$ and $\eta=\sqrt{\mu_0/\epsilon_0}$.

Next, consider the radiation from an open-end waveguide that is completely filled with ferrite which is transversely magnetized. We shall solve this problem for a multimode waveguide, *i.e.*, a waveguide which can propagate more than one mode. The geometry of the problem is shown in Fig. 6. The far-zone electric field patterns are found to be the following: in the H plane ($\phi=0$)

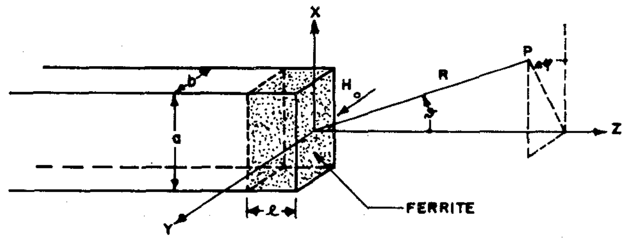

Fig. 6—Geometry of the problem of radiation from an open-end waveguide containing ferrite.

effect of the magnetostatic field H_0 upon the shape and the relative positions of the E_ϕ and E_θ field patterns. To do this we assume the following parameters:

Ferrite saturation magnetization—$p=0.6$
Ferrite dielectric constant[18]—$\epsilon'=1$
Thickness of the ferrite plate—$l=2$ cm
Carrier wavelength—$\lambda=3.2$ cm
Waveguide inside dimensions—$\begin{cases} a=2.28 \text{ cm} \\ b=1.02 \text{ cm}\end{cases}$
Mode conversion ratio—$C_2/C_1=1$.

$$E_\phi = \frac{j\pi ab}{4\lambda R}C_1 e^{-j(kR+\beta_1 l+\xi_1-u)}\sum_{m=1}^{M}\left\{\frac{mC_m}{C_1}\left\{\frac{\left|\sin\left(u+\frac{m\pi}{2}\right)\right|}{\left(\frac{m\pi}{2}\right)^2-u^2}\right\}\right.$$

$$\left.\text{times}\sqrt{\left(\cos\theta+\frac{\beta_m\lambda}{2\pi\chi}\right)^2+\left(\frac{\lambda}{\pi\chi}\cdot\frac{\kappa}{\mu}\cdot\frac{u}{a}\right)^2}\, e^{j[(\beta_1-\beta_m)l+(\xi-\xi_m)]}\right\}, \quad (62)$$

and in the E plane ($\varphi=\pi/2$)

$$E_\theta = \frac{ab}{\pi\lambda R}C_1\frac{\sin u'}{u'}e^{-j(kR+\beta_1 l-u')}$$

$$\cdot\sum_{m=1}^{M}\frac{C_m}{mC_1}\sin\frac{m\pi}{2}\left(1+\frac{\lambda\beta_m\cos\theta}{2\pi\chi}\right)e^{j(\beta_1-\beta_m)l} \quad (63)$$

where

$$\xi_m = \arctan\left[\left\{\frac{\cos\theta+\frac{\lambda\beta_m}{2\pi\chi}}{\frac{\lambda}{\pi\chi}\cdot\frac{\kappa}{\mu}\cdot\frac{u}{a}}\right\}^{(-1)^m}\right] \quad (64)$$

$$u = \frac{\pi a}{\lambda}\sin\theta; \qquad u' = \frac{\pi b}{\lambda}\sin\theta. \quad (65)$$

The implications of the above radiation field patterns are not easy to perceive. It will be advantageous at this point to apply these equations to a specific numerical example which would enable us to get a better insight into the situation. Specifically, we are interested in the

It can be seen from Fig. 3 that the TE_{10} mode will propagate through the waveguide when $0.0\le\sigma<0.15$ and when $\sigma>0.85$ and it will be cut off when $0.15<\sigma<0.85$. The TE_{20} mode will propagate when $0.85<\sigma<1.17$. The TE_{30} mode is cut off completely except for a very small region ($0.85<\sigma<0.90$) which we neglect.

The aperture electric field distribution, E_y, for the two-mode transmission and the phase angle ψ are plotted for several values of the external magnetostatic field H_0 in Fig. 7. It can be seen that the field is shifted to the left-hand side of the guide at $\sigma=0.85$. As the magnetostatic field is increased, the field gradually shifts back to the center of the guide with a decrease in magnitude and it is symmetrical with respect to the center of the guide somewhere between $\sigma=1.05$ and $\sigma=1.10$. As the magnetostatic field is further increased, the electric field shifts to the right-hand side of the guide

[18] It is realized that a dielectric constant of one is not realistic for a ferrite of today since the ferrites produced nowadays have a dielectric constant of about ten. The value of one was introduced here merely to simplify the many numerical calculations that had to be carried out. It is true that when $\epsilon=10$ not only may two modes propagate, but a great many. One may expect, however, that the amplitudes of the higher order modes will be progressively smaller so that their effect may be neglected to the first order of approximation.

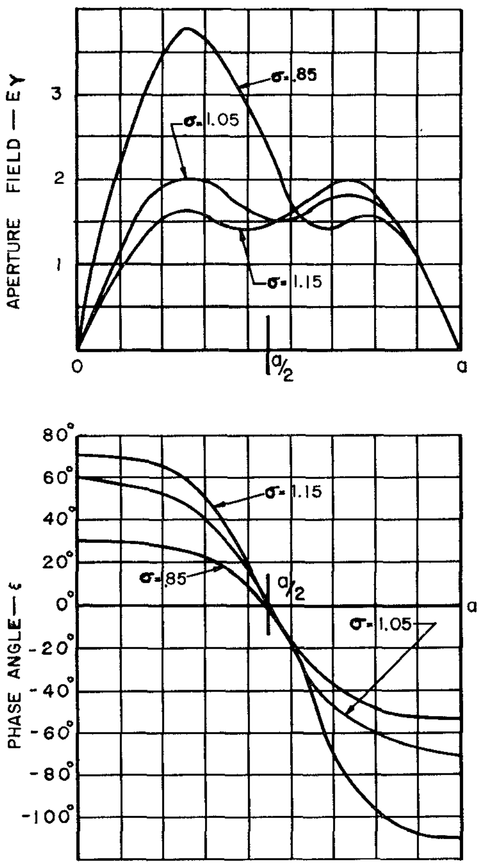

Fig. 7—Aperture electric field and phase angle.

and at $\sigma = 1.15$ it is exactly opposite with respect to the center of the guide to the position it had at $\sigma = 0.85$.

The electric field pattern in the H plane is plotted in Fig. 8 for a range of magnetostatic field from $\sigma = 0.0$ to $\sigma = 2.0$. Starting with $\sigma = 0.0$ the amplitude of the pattern gradually decreases and it becomes zero at $\sigma = 0.15$, where the propagation constant for the principal mode becomes zero (Fig. 3). There is no radiation between $\sigma = 0.15$ to $\sigma = 0.84$ since all modes in the guide are cut off. At $\sigma = 0.85$ transmission through the guide begins again. Now the waveguide is large enough electrically so that not only the principal mode but also the second-order mode can propagate freely; the radiation pattern is large in magnitude and it reaches its maximum at about -25 degrees off the zero axis. As the magnetostatic field is increased the radiation pattern decreases slightly in amplitude and it shifts farther in the clockwise direction. At the same time, it can be seen, a new sidelobe appears at $+70$ degrees. As the magnetostatic field is further increased, the main lobe remains in the same position as before and the new lobe gradually builds up and moves closer toward the center. This process of contraction of the previous main lobe and the expansion of the new sidelobe on the opposite side goes on until, at $\sigma = 1.15$, a new main lobe is definitely formed at $+20$ degrees. As the magnetostatic field is further increased, the waveguide decreases

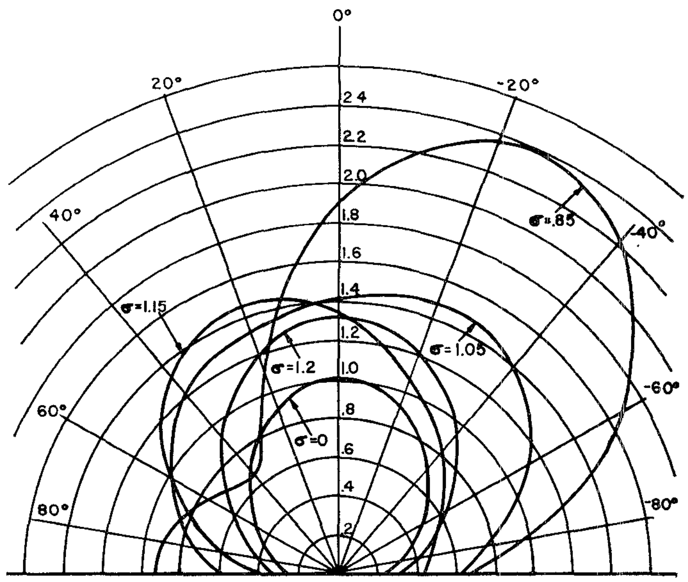

Fig. 8—H-plane electric field pattern.

in size electrically so that the second-order mode is cut off and only the principal one remains. The radiation pattern is now centered again about the zero axis and further increase in the magnetization changes the magnitude only.

The process of pattern formation may be best explained by reference to Fig. 9. The components E_ϕ' and E_ϕ'', corresponding to the radiation patterns of the first and second modes which make up E_ϕ, are plotted there together with the phase angle Γ_2 for a constant value of magnetic fieid, σ. Since E_ϕ', E_ϕ'', and Γ_2 are all functions of the spherical angle θ and the magnetic field σ, they add in various combinations of magnitudes and phase angle to form the final pattern, E_ϕ.

It may be of interest to estimate the value of error that might have been introduced here due to the assumption that the ratio of mode amplitudes $C_2/C_1 = 1$. The value of this ratio affects the pattern by the way of changing the magnitude of E_ϕ'' in (62). In Fig. 10 three electric field patterns are plotted which correspond to the values of C_2/C_1 of 1, 2, and 0.5. Taking $C_2/C_1 = 1$ pattern as a reference, it may be seen that pattern deviations are significant for values of $C_2/C_1 > 1$, and are relatively small for values of $C_2/C_1 < 1$. However, in each case the pattern is still shifted in the same direction as before. Intuitively, we could say that C_2/C_1 should be smaller than one which concludes that the patterns in Fig. 8 are in no great error, due to the assumption $C_2/C_1 = 1$.

The patterns we have just discussed are transmitting patterns. One may ask whether the same patterns are receiving patterns, too. It was shown in (56) and (57) that the wave admittances of the incident and reflected waves in the aperture are identical except for the sign of κ. It was also shown in Fig. 5 that the input and output wave impedances had a mirror symmetry about the

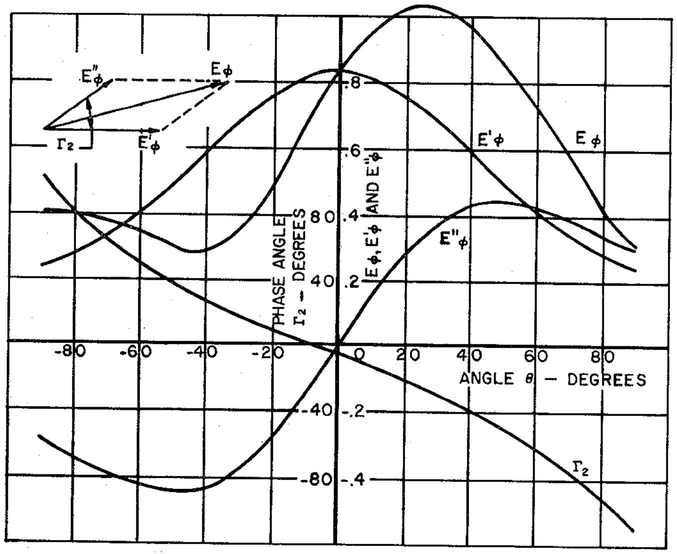

Fig. 9—The mechanics of pattern formation.

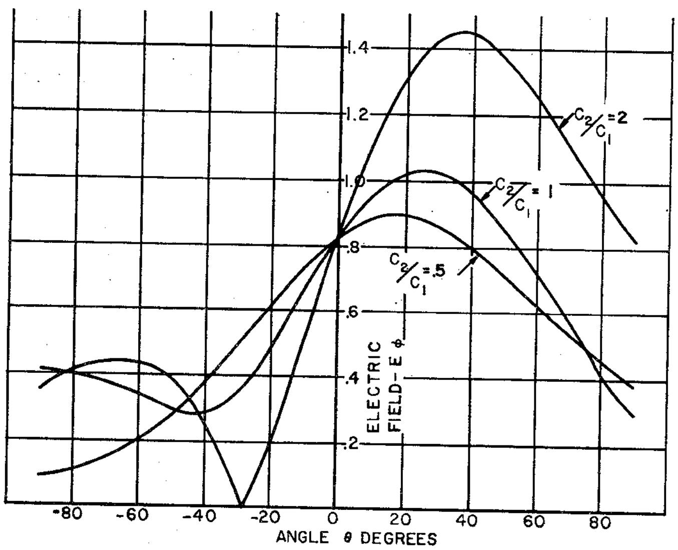

Fig. 10—The effect of the mode conversion ratio C_2/C_1 upon the shape of the pattern.

center of the guide. Thus, from the argument just presented, one may conclude that the transmitting and the receiving patterns, which are functions of the output and input impedance, respectively, will also have a mirror symmetry (see Fig. 11). Also, since the reversal of the direction of the magnetostatic field changes the sign of κ, the receiving and transmitting patterns are interchanged when the magnetostatic field is reversed. These phenomena were demonstrated experimentally by Angelakos and Korman.[1] Their results are reproduced in Fig. 12 for comparison.[19]

The equations for the far-zone radiation fields from an open-end waveguide filled with ferrite magnetized in the direction of propagation are derived in the Appendix.

[19] Angelakos and Korman, *op. cit.*, Fig. 12, p. 1468.

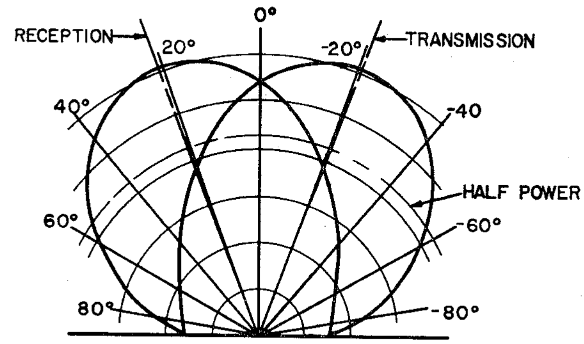

Fig. 11—Nonreciprocity of transmitting and receiving patterns.

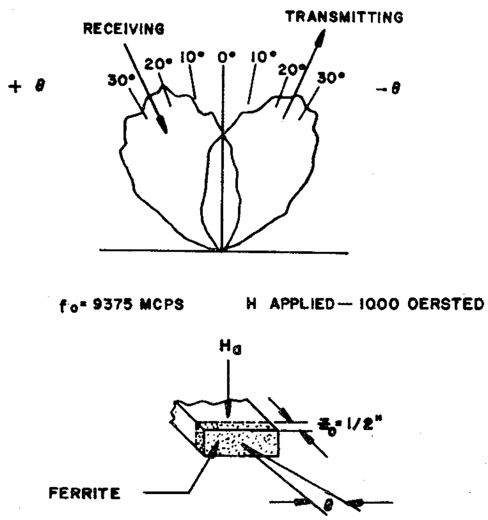

Fig. 12—Experimental patterns of Angelakos and Korman. (Reproduced by permission of the authors.)

Appendix

Radiation from an Open-End Waveguide Filled with Ferrite Longitudinally Magnetized

In this case we restrict ourselves to a waveguide which can propagate a single TE mode only. The geometry of the problem is similar to that of Fig. 6 except that now the direction of the magnetostatic field is in the direction of propagation.

Substituting the expressions for the transverse field components found in (26) through (29) into (58) through (61) and carrying out the integration we obtain: in the H plane ($\phi=0$)

$$E_\theta = \frac{j^m ab\pi \sin\left(u+\frac{m\pi}{2}\right)}{4\lambda R} \cdot \left\{ \frac{mA_1}{\left(\frac{m\pi}{2}\right)^2 - u^2} - \frac{nA_2}{\left(\frac{n\pi}{2}\right)^2 - u^2} \right\} e^{-j(kR-u)} \tag{66}$$

$$E_\phi = \frac{-j^m ab\pi \sin\left(u+\frac{m\pi}{2}\right)}{4\lambda R}$$

$$\cdot \left\{ \frac{mA_3}{\left(\frac{m\pi}{2}\right)^2 - u^2} - \frac{nA_4}{\left(\frac{n\pi}{2}\right)^2 - u^2} \right\} e^{-j(kR-u)}, \quad (67)$$

and in the E plane ($\phi = \pi/2$)

$$E_\theta = \frac{-j^m \sin\left(\frac{m\pi}{2}\right) ab}{\lambda\pi R} \left\{ \frac{A_5}{m} - \frac{A_6}{n} \right\} \frac{\sin u'}{u'} e^{-j(kR-u')} \quad (68)$$

$$E_\phi = \frac{-j^m \sin\left(\frac{m\pi}{2}\right) ab}{\lambda\pi R} \left\{ \frac{A_7}{m} - \frac{A_8}{n} \right\} \frac{\sin u'}{u'} e^{-j(kR-u')} \quad (69)$$

where

$$A_1 = j \frac{Ca}{m\pi} (\beta + \omega\epsilon\eta \cos\theta) \quad (70)$$

$$A_2 = j \frac{Ca}{n\pi} (\beta + \omega\epsilon\eta \cos\theta) \quad (71)$$

$$A_3 = \frac{Ca}{m\pi\kappa} \left\{ \frac{\eta}{\omega} \left[\left(\frac{m\pi}{2}\right)^2 - N \right] + \frac{\mu \cos\theta}{\beta} \left[\left(\frac{m\pi}{a}\right)^2 - M \right] \right\} \quad (72)$$

$$A_4 = \frac{Ca}{n\pi\kappa} \left\{ \frac{\eta}{\omega} \left[\left(\frac{n\pi}{a}\right)^2 - N \right] + \frac{\mu \cos\theta}{\beta} \left[\left(\frac{n\pi}{a}\right)^2 - M \right] \right\} \quad (73)$$

$$A_5 = \frac{Ca}{m\pi\kappa} \left\{ \frac{\mu}{\beta} \left[\left(\frac{m\pi}{a}\right)^2 - M \right] + \frac{\eta \cos\theta}{\omega} \left[\left(\frac{m\pi}{a}\right)^2 - N \right] \right\} \quad (74)$$

$$A_6 = \frac{Ca}{m\pi\kappa} \left\{ \frac{\mu}{\beta} \left[\left(\frac{n\pi}{a}\right)^2 - M \right] + \frac{\eta \cos\theta}{\omega} \left[\left(\frac{n\pi}{a}\right)^2 - N \right] \right\} \quad (75)$$

$$A_7 = \frac{jCa}{m\pi} (\beta \cos\theta + \eta\omega\epsilon) \quad (76)$$

$$A_8 = \frac{jCa}{n\pi} (\beta \cos\theta + \eta\omega\epsilon). \quad (77)$$

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Launching Efficiency of Wires and Slots for a Dielectric Rod Waveguide*

R. H. DuHAMEL† AND J. W. DUNCAN‡

Summary—**This paper describes an experimental investigation of surface wave launching efficiency. Wires, rings, and slots are considered as exciters of the HE_{11} mode on a dielectric rod image line. A formula is derived which relates the efficiency of a launcher to its impedance as a scatterer on the surface waveguide. Efficiency is obtained by using this formula and also by applying Deschamps' method for determining the scattering matrix coefficients of a two-port junction. Graphs are presented which illustrate the variation of efficiency with the dimensions of the launchers and with the parameter λ_g/λ, the ratio of the guide wavelength to the free space wavelength.**

Introduction

THE THEORY of wave propagation on dielectric rods has been treated extensively by a number of investigators [1]–[3]. In recent years the dielectric rod waveguide has been employed with considerable success as a dielectric antenna [4], [5]. The mode which is most often used for dielectric rod antennas is the HE_{11} (or dipole) mode. It is the lowest order mode which can propagate on a dielectric rod and has no

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